# SOME BINOMIAL SERIES OBTAINED BY THE WZ-METHOD

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Abstract. Using the WZ-method we find some of the easiest Ramanujan's formulae and also some new interesting Ramanujan-like sums.

## 1. The WZ-method

We recall [5] that a discrete function $A(n,k)$ is hypergeometric or closed form (CF) if the quotients

$$\frac{A(n+1,k)}{A(n,k)} \quad \text{and} \quad \frac{A(n,k+1)}{A(n,k)}$$

are both rational functions.

And a pair of functions $F(n,k)$, $G(n,k)$ is said to be of Wilf and Zeilberger (WZ) if $F$ and $G$ are closed forms and besides

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k).$$

In this case H. S. Wilf and D. Zeilberger [4] have proved that there exists a rational function $C(n,k)$ such that

$$G(n,k) = C(n,k)F(n,k).$$

The rational function $C(n,k)$ is the so-called certificate of the pair $(F,G)$.

We now define

$$H(n,k) = F(n+1,n+k) + G(n,n+k).$$

Zeilberger has proved that for every WZ pair $F(n,k)$, $G(n,k)$ the following holds:

$$\sum_{n=0}^{\infty} G(n,0) = \sum_{n=0}^{\infty} H(n,0).$$

In next sections we use WZ-pairs to get some Ramanujan's formulae and also some new Ramanujan-like ones.

## 2. First WZ-pair

We consider the following discrete function:

$$G(n,k) = \frac{(-1)^n(-1)^k}{2^{10n}2^{2k}}(20n+2k+3)\frac{\binom{2k}{k}^2\binom{2n}{n}^2\binom{4n-2k}{2n-k}}{\binom{2n}{k}\binom{n+k}{n}}.$$

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The package EKHAD [2] allows to obtain the companion

$$F(n,k) = 64 \frac{(-1)^n (-1)^k}{2^{10n} 2^{2k}} \frac{n^2}{4n-2k-1} \frac{\binom{2k}{k}^2 \binom{2n}{n}^2 \binom{4n-2k}{2n-k}}{\binom{2n}{k}\binom{n+k}{n}}.$$

We derive the result

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{4n}{2n}\binom{2n}{n}^2}{2^{10n}} (20n+3) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^3}{2^{12n}} (42n+5).$$

We can extend the pair to have sense for every value of $k$, not only integers, in the following way:

$$F(n,k) = \frac{64}{\pi^3} \frac{n^2}{4n-2k-1} \frac{(-1)^n \cos(\pi k)\Gamma(2n-k+1/2)\Gamma(n+1/2)^3\Gamma(k+1/2)^2}{\Gamma(n+k+1)\Gamma(2n+1)^2},$$

$$G(n,k) = \frac{1}{\pi^3}(20n+2k+3) \frac{(-1)^n \cos(\pi k)\Gamma(2n-k+1/2)\Gamma(n+1/2)^3\Gamma(k+1/2)^2}{\Gamma(n+k+1)\Gamma(2n+1)^2}.$$

If $k$ is an integer, it is a routine to prove that

$$\sum_{n=0}^{\infty} G(n,k) = \sum_{n=0}^{\infty} G(n,k+1),$$

and this implies applying Carlson's theorem [1] that for every value of $k$, even if $k$ is not an integer, $\sum_{n=0}^{\infty} G(n,k) = A$, where $A$ is a constant. To determine the value of the constant, observe that

$$\lim_{t \to 1/2} \sum_{n=1}^{\infty} G(n,t) = 0 \quad \Rightarrow \quad A = \lim_{t \to 1/2} G(0,t) = \frac{8}{\pi}.$$

And we have that, independently of the value of $k$,

$$\sum_{n=0}^{\infty} G(n,k) = \frac{8}{\pi}.$$

But then we have also the sum of another family of infinite series because obviously we immediately get

$$\sum_{n=0}^{\infty} H(n,k) = \frac{8}{\pi}.$$

For $k=0$, we get the following results [3]:

$$\sum_{n=0}^{\infty} G(n,0) = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{4n}{2n}\binom{2n}{n}^2}{2^{10n}} (20n+3) = \frac{8}{\pi},$$

$$\sum_{n=0}^{\infty} H(n,0) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^3}{2^{12n}} (42n+5) = \frac{8}{\pi}.$$

For other values of $k$ we obtain also interesting results. For example, for $k = 1/4$ we get

$$\frac{\sqrt{2}}{8}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{2}\right)_n\left(\frac{1}{4}\right)_{2n}}{(n!)^2\left(\frac{1}{4}\right)_n 2^{4n}}\frac{40n+7}{4n+1}=\frac{\sqrt{\pi}}{\Gamma(\frac{3}{4})^2},$$

$$\frac{3\sqrt{2}}{8}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_{2n}^2\left(\frac{1}{2}\right)_n}{(n!)^2\left(\frac{1}{4}\right)_{2n}\left(\frac{1}{4}\right)_n 2^{8n}}\frac{112n^2+88n+11}{(8n+1)(8n+5)}=\frac{\sqrt{\pi}}{\Gamma(\frac{3}{4})^2}.$$

## 3. Second WZ-pair

We consider the following discrete function:

$$G(n,k)=\frac{(-1)^k}{2^{16n}2^{4k}}(120n^2+84nk+34n+10k+3)\frac{\binom{2k}{k}^3\binom{2n}{n}^4\binom{4n-2k}{2n-k}}{\binom{2n}{k}\binom{n+k}{n}^2};$$

the package EKHAD [2] allows to get the companion

$$F(n,k)=512\frac{(-1)^k}{2^{16n}2^{4k}}\frac{n^3}{4n-2k-1}\frac{\binom{2k}{k}^3\binom{2n}{n}^4\binom{4n-2k}{2n-k}}{\binom{2n}{k}\binom{n+k}{n}^2}.$$

We have the following result:

$$\sum_{n=0}^{\infty}\frac{\binom{4n}{2n}\binom{2n}{n}^4}{2^{16n}}(120n^2+34n+3)=\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n}{n}^5}{2^{20n}}(820n^2+180n+13).$$

We can extend the pair to have sense for every value of $k$, not only integers, in the following way:

$$F(n,k)=\frac{512}{\pi^5}\frac{n^3}{4n-2k-1}\times\frac{\cos(\pi k)\Gamma(2n-k+1/2)\Gamma(n+1/2)^6\Gamma(k+1/2)^3}{\Gamma(n+k+1)^2\Gamma(2n+1)^3},$$

$$G(n,k)=\frac{1}{\pi^5}(120n^2+84nk+34n+10k+3)\times\frac{\cos(\pi k)\Gamma(2n-k+1/2)\Gamma(n+1/2)^6\Gamma(k+1/2)^3}{\Gamma(n+k+1)^2\Gamma(2n+1)^3}.$$

If $k$ is an integer it is a routine to prove that

$$\sum_{n=0}^{\infty}G(n,k)=\sum_{n=0}^{\infty}G(n,k+1),$$

and this implies applying Carlson's theorem [1] that for every value of $k$, even if $k$ is not an integer, $\sum_{n=0}^{\infty}G(n,k)=A$, where $A$ is a constant. To determine the constant value $A$ observe that

$$\lim_{t\to 1/2}\sum_{n=1}^{\infty}G(n,t)=0 \quad\Rightarrow\quad A=\lim_{t\to 1/2}G(0,t)=\frac{32}{\pi^2}.$$

And we have that, independently of the value of $k$,

$$\sum_{n=0}^{\infty}G(n,k)=\frac{32}{\pi^2}.$$

But then we have also the sum of another family of infinite series because obviously we immediately get

$$\sum_{n=0}^{\infty} H(n,k) = \frac{32}{\pi^2}.$$

For $k=0$ we obtain the following results:

$$\sum_{n=0}^{\infty} G(n,0) = \sum_{n=0}^{\infty} \frac{\binom{4n}{2n}\binom{2n}{n}^4}{2^{16n}}(120n^2+34n+3) = \frac{32}{\pi^2},$$

$$\sum_{n=0}^{\infty} H(n,0) = \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n\binom{2n}{n}^5}{2^{20n}}(820n^2+180n+13) = \frac{32}{\pi^2}.$$

For other values of $k$ we obtain also interesting results. For example, for $k=1/4$ we get

$$\frac{1}{8}\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}\right)_{2n}}{(n!)^3 \left(\frac{1}{4}\right)_n^2 2^{6n}} \frac{240n^2+110n+11}{(4n+1)^2} = \frac{\pi}{\Gamma(\frac{3}{4})^4},$$

$$\frac{1}{8}\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_{2n}^3 \left(\frac{1}{2}\right)_n^3}{(n!)^3 \left(\frac{1}{4}\right)_{2n}^2 \left(\frac{1}{4}\right)_n^2 2^{12n}} \frac{26240n^4+41184n^3+21448n^2+4170n+279}{(8n+1)^2(8n+5)^2} = \frac{\pi}{\Gamma(\frac{3}{4})^4}.$$

## References

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