GENERATORS OF SOME RAMANUJAN'S FORMULAS

JESÚS GUILLERA

Abstract. In this paper we prove some Ramanujan-type formulas for $1/\pi$ but without using the theory of modular forms. Instead we use the WZ-method created by H. Wilf and D. Zeilberger and find some hypergeometric functions in two variables which are second components of WZ-pairs as can be certified by the package EKHAD, these certificates have an additional property which allows us to get generalized Ramanujan-type series which are routinely proven by computer. We call these second hypergeometric components of the WZ-pairs generators. Finding generators seems a hard task but using a kind of experimental research (explained below), we have succeeded in finding some of them. Unfortunately we have not yet found generators for the most impressive Ramanujan's formulas. We also prove some interesting binomial sums for the constant $1/\pi^2$ Finally we rewrite many of the obtained series using pochhammer symbols and study the rate of convergence.

Some generators

We consider the functions in TABLE 1, (generators of first order). With package EKHAD [4] we can check that all these hypergeometric functions $G_1(n,k)$ are second components of WZ-pairs [7] and at the same time get the first components $F_1(n,k) = C_1(n,k)G_1(n,k)$, where $C_1(n,k)$ is the certificate [6]. We know that $F_1(n,k)$ and $G_1(n,k)$ are related by

$$F_1(n+1,k) - F_1(n,k) = G_1(n,k+1) - G_1(n,k)$$

But in each case considered we have $F_1(0,k) = 0$ and so

$$\sum_{n=0}^{\infty} [G_1(n,k+1 - G_1(n,k)] = \sum_{n=0}^{\infty} [F_1(n+1,k) - F_1(n,k)] = -F_1(0,k) = 0$$

which allows us to write

$$\sum_{n=0}^{\infty} G_1(n,0) = \sum_{n=0}^{\infty} G_1(n,1) = \sum_{n=0}^{\infty} G_1(n,2) = \sum_{n=0}^{\infty} G_1(n,3) = \cdots = cte$$

Even more, if we make the substitution of $(-1)^k$ by $\cos \pi k$ the conditions of Carlson's theorem [1] hold and applying it we conclude that for every real number k we have

$$\sum_{n=0}^{\infty} G_1(n,k) = cte$$

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For the functions $G_1(n,k)$ we are using we can get the value of the constant by taking the limit at $k=1/2$. And some Ramanujan's formulas are obtained by plugging $k=0$. If we now define the function, (generators of second order)

$$G_2(n,k) = F_1(n+1,n+k) + G_1(n,n+k)$$

then from Zeilberger's theorem we can easily derive that

$$\sum_{n=0}^{\infty} G_2(n,k) = \sum_{n=0}^{\infty} G_1(n,k) = cte$$

and again some Ramanujan's formulas appear by plugging $k=0$. We show these formulas in TABLE 2 [5] We can also define the generators of higher order and of course we have

$$\sum_{n=0}^{\infty} G_1(n,k) = \sum_{n=0}^{\infty} G_2(n,k) = \sum_{n=0}^{\infty} G_3(n,k) = \sum_{n=0}^{\infty} G_4(n,k) = \cdots = cte$$

Observation: With the indicated generators not only have we succeeded in proving some Ramanujan's formulas in an easy way, but also obtained many new interesting formulas. An equivalent to the first generator was given by D. Zeilberger [2] as an example to show this new way of proving Ramanujan's formulas.

We now show explicitly the most interesting generalized series we have found classified according to the type of binomials they use in case $k=0$.

First group of formulas

$$\sum_{n=0}^{\infty} \frac{(\cos \pi k)(-1)^n}{2^{6n}2^{2k}} \frac{\binom{2n}{n}^3 \binom{2k}{k}^2}{2^{2k}\binom{n-1/2}{k}\binom{n+k}{n}} (4n+1) = \frac{2}{\pi}$$

$$\sum_{n=0}^{\infty} \frac{(\cos \pi k)}{2^{8n}2^{2k}} \frac{\binom{2n}{n}^2 \binom{2k}{k}\binom{2n+2k}{n+k}}{2^{2k}\binom{n-1/2}{k}} (6n+2k+1) = \frac{4}{\pi}$$

$$\sum_{n=0}^{\infty} \frac{(\cos \pi k)(-1)^n}{2^{9n}2^{3k}} \frac{\binom{2n}{n}\binom{n+k}{n}\binom{2n+2k}{n+k}^2}{2^{2k}\binom{n-1/2}{k}} (6n+2k+1) = \frac{2\sqrt{2}}{\pi}$$

$$\sum_{n=0}^{\infty} \frac{(\cos \pi k)(-1)^n}{2^{12n}2^{2k}} \frac{\binom{2n}{n}^2\binom{2n+2k}{n+k}^2\binom{n+k}{n}}{2^{2k}\binom{n-1/2}{k}\binom{2n+k}{n}} \frac{84n^2+56nk+52n+4k^2+12k+5}{2n+k+1} = \frac{16}{\pi}$$

Second group of formulas

$$\sum_{n=0}^{\infty} \frac{(\cos \pi k)(-1)^n}{2^{10n}2^{2k}} \frac{\binom{2n}{n}^2\binom{2k}{k}^2\binom{4n}{2n}}{2^{2k}\binom{2n-1/2}{k}\binom{n+k}{n}} (20n+2k+3) = \frac{8}{\pi}$$

$$\sum_{n=0}^{\infty} \frac{(\cos \pi k)3^k}{2^{8n}2^{4k}3^{2n}} \frac{\binom{2n}{n}\binom{4n}{2n}\binom{2k}{k}\binom{2n+2k}{n+k}}{2^{2k}\binom{2n-1/2}{k}} (8n+2k+1) = \frac{2\sqrt{3}}{\pi}$$

$$\sum_{n=0}^{\infty} \frac{(\cos \pi k)(-1)^n 3^k}{2^{12n}3^n 2^{4k}} \frac{\binom{2n}{n}\binom{2n+2k}{n+k}\binom{4n+2k}{2n+k}\binom{n+k}{n}}{2^{2k}\binom{n-1/2}{k}} \frac{56n^2+36nk+34n+4k^2+8k+3}{2n+k+1} = \frac{16\sqrt{3}}{3\pi}$$

Experimental method for finding generators

As an example I am going to explain how I found the formula

$$\sum_{n=0}^{\infty} \frac{(\cos \pi k) 3^k}{2^{8n} 2^{4k} 3^{2n}} \frac{\binom{2n}{n}\binom{4n}{2n}\binom{2k}{k}\binom{2n+2k}{n+k}}{2^{2k}\binom{2n-1/2}{k}} (8n+2k+1) = \frac{2\sqrt{3}}{\pi}$$

We consider the function

$$B(n,k) = \frac{\binom{2n+2k}{n+k}^p}{\binom{2n}{n}^p} \frac{\binom{4n+2k}{2n+k}^q}{\binom{4n}{2n}^q} \binom{n+k}{n}^r \frac{1}{\binom{2n-1/2}{k}} \binom{4n}{2n}\binom{2n}{n}^2$$

Observe that for $k=0$ we have

$$B(n,0) = \binom{4n}{2n}\binom{2n}{n}^2$$

We are looking for a generator of the type

$$G(n,k) = f(k) \frac{B(n,k)}{2^{8n} 3^{2n}} (an + bk + c)$$

such that

$$\sum_{n=0}^{\infty} G(n,k) = \sum_{n=0}^{\infty} G(n,0) = \frac{2\sqrt{3}}{\pi}$$

We can solve this problem in an experimental way looking for integer relations between

$$\frac{\sqrt{3}}{\pi}, \quad \sum_{n=0}^{\infty} B(n,k), \quad \sum_{n=0}^{\infty} B(n,k) n.$$

The structure of program is like this

$$FOR \quad p = -3 \quad TO \quad 3$$
$$FOR \quad q = -3 \quad TO \quad 3$$
$$FOR \quad r = -3 \quad TO \quad 3$$
$$\vdots$$
$$u(k) = \quad INTEGER \quad RELATION \left(\frac{\sqrt{3}}{\pi}, \quad \sum_{n=0}^{\infty} B(n,k), \quad \sum_{n=0}^{\infty} B(n,k) n \right)$$

I have written a code in PARI-GP following the above idea.

For $k=0$ we have, of course, the corresponding Ramanujan's formula. So we must investigate $u(1), u(2)$, etc. For these computational purposes we use the following simplifications

$$\frac{\binom{2n+2k}{n+k}}{\binom{2n}{n}} = \frac{\prod_{j=1}^{2k}(2n+j)}{\prod_{j=1}^{k}(n+j)^2} \qquad \frac{\binom{4n+2k}{2n+k}}{\binom{4n}{2n}} = \frac{\prod_{j=1}^{2k}(4n+j)}{\prod_{j=1}^{k}(2n+j)^2}$$

$$\binom{n+k}{k} = \frac{1}{k!}\prod_{j=1}^{k}(n+j) \qquad \binom{2n-1/2}{k} = \frac{1}{k!}\prod_{j=1}^{k}(2n+1/2-j)$$

If there are still integer relations then we know the function $B(n,k)$ is valid and we can easily find the values of a, b, and c. So, it only rests to know the function $f(k)$, but using the package EKHAD this task is very simple.

A new kind of Ramanujan's formulas

I show some examples of a new kind of ramanujan's type formulas [3]

$$\sum_{n=0}^{\infty} \frac{(\cos \pi k)(-1)^n}{2^{12n} 2^{4k}} \frac{\binom{2n}{n}^4 \binom{2n+2k}{n+k} \binom{2k}{k}^2}{2^{2k} \binom{n-1/2}{k} \binom{n+k}{n}} (20n^2 + 12kn + 8n + 2k + 1) = \frac{8}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\cos \pi k}{2^{16n} 2^{4k}} \frac{\binom{2n}{n}^4 \binom{4n}{2n} \binom{2k}{k}^3}{2^{2k} \binom{2n-1/2}{k} [\binom{n+k}{n}^2} (120n^2 + 84kn + 34n + 10k + 3) = \frac{32}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{(\cos \pi k)(-1)^n}{2^{20n} 2^{4k}} \frac{\binom{2n}{n}^4 \binom{2n+2k}{n+k}^3 \binom{n+k}{n}}{2^{2k} \binom{n-1/2}{k} \binom{2n+k}{n}^2} \frac{P(n,k)}{(2n+k+1)^2} = \frac{128}{\pi^2},$$

where

$$P(n,k) = 3280n^4 + 4000n^3 + 1592n^2 + 232n + 13 + 4592kn^3 + 3816kn^2 + \\ 884nk + 62k + +1008nk^2 + 336nk^3 + 2160n^2k^2 + 92k^2 + 40k^3.$$

For $k = 0$ we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}^5}{2^{12n}} (20n^2 + 8n + 1) = \frac{8}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^4 \binom{4n}{2n}}{2^{16n}} (120n^2 + 34n + 3) = \frac{32}{\pi^2},$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}^5}{2^{20n}} (820n^2 + 180n + 13) = \frac{128}{\pi^2}.$$

The last one adds roughly three digits per term.

Some more series

we get these sums by comparing them to other series in this paper at the value $k = 0$. It is also interesting to evaluate them at $k = 1/2$.

$$\sum_{n=0}^{\infty} \frac{1}{2^{8n} 2^{4k}} \frac{\binom{2n}{n}^3 \binom{2k}{k}^2}{\binom{n+k}{n}^2} (6n + 4k + 1) = \frac{4}{\pi}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{6n} 2^{4k}} \frac{\binom{2n}{n}^2 \binom{2n+2k}{n+k} \binom{2k}{k}}{\binom{n+k}{n}} (4n + 2k + 1) = \frac{2}{\pi}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{12n} 2^{8k}} \frac{\binom{2n}{n}^5 \binom{2k}{k}^4}{\binom{n+k}{n}^4} (20n^2 + 8n + 1 + 24kn + 8k^2 + 4k) = \frac{8}{\pi^2}$$

Rate of convergence

To study the rate of convergence of the series in this paper and also for other purposes, like getting neat sums when k is a rational number, it is convenient to express the sums in an equivalent form using pochhammer symbols. By doing it to many of the obtained series we have formulas with the following aspect

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}-k\right)_n \left(\frac{1}{2}+k\right)_n}{n!^2(1+k)_n}(6n+2k+1) = \frac{4}{\pi} \frac{2^{2k}}{\binom{2k}{k}}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{3n}} \frac{\left(\frac{1}{2}-k\right)_n \left(\frac{1}{2}+k\right)_n \left(\frac{1}{2}+k\right)_n}{n!^2(1+k)_n}(6n+2k+1) = \frac{2\sqrt{2}}{\pi} \frac{2^{3k}}{\binom{2k}{k}}$$

$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}-k\right)_n \left(\frac{1}{2}+k\right)_n}{n!^2\left(1+\frac{k}{2}\right)_n} \frac{\left(\frac{1}{2}+k\right)_n}{\left(\frac{1}{2}+\frac{k}{2}\right)_n} R(n,k) = \frac{16}{\pi} \frac{2^{2k}}{\binom{2k}{k}}$$

where

$$R(n,k) = \frac{84n^2+56nk+52n+4k^2+12k+5}{2n+k+1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}-\frac{k}{2}\right)_n \left(\frac{3}{4}-\frac{k}{2}\right)_n}{n!^2(1+k)_n}(20n+2k+3) = \frac{8}{\pi} \frac{2^{2k}}{\binom{2k}{k}}$$

$$\sum_{n=0}^{\infty} \frac{1}{3^{2n}} \frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{4}-\frac{k}{2}\right)_n \left(\frac{3}{4}-\frac{k}{2}\right)_n}{n!^2(1+k)_n}(8n+2k+1) = \frac{2\sqrt{3}}{\pi} \frac{2^{4k}}{3^k\binom{2k}{k}}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n}3^n} \frac{\left(\frac{1}{2}-k\right)_n \left(\frac{1}{4}+\frac{k}{2}\right)_n \left(\frac{3}{4}+\frac{k}{2}\right)_n}{n!^2\left(1+\frac{k}{2}\right)_n} \frac{\left(\frac{1}{2}+k\right)_n}{\left(\frac{1}{2}+\frac{k}{2}\right)_n} R(n,k) = \frac{16\sqrt{3}}{3\pi} \frac{2^{4k}}{3^k\binom{2k}{k}}$$

where

$$R(n,k) = \frac{56n^2+36nk+34n+4k^2+8k+3}{2n+k+1}$$

The following two formulas are sums of hypergeometric series of type 5×4

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{2}-k\right)_n \left(\frac{1}{2}+k\right)_n}{n!^3(1+k)_n^2}(20n^2+12kn+8n+2k+1) = \frac{8}{\pi^2} \frac{2^{4k}}{\binom{2k}{k}^2}$$

$$\sum_{n=0}^{\infty} \frac{1}{2^{4n}} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}-\frac{k}{2}\right)_n \left(\frac{3}{4}-\frac{k}{2}\right)_n}{n!^3(1+k)_n^2}(120n^2+84kn+34n+10k+3) = \frac{32}{\pi^2} \frac{2^{4k}}{\binom{2k}{k}^2}$$

The rate of convergence is roughly the same of the geometric series obtained taking just the first quotients, in the sense that the number of exact digits they give is asymptotically the same.

Now I am going to consider one of the above series and show how to get a more accurate behavior of its rate of convergence. The example I choose is

$$\sum_{n=0}^{\infty} a(k,n) = \sum_{n=0}^{\infty} \frac{1}{2^{4n}} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{4}-\frac{k}{2}\right)_n \left(\frac{3}{4}-\frac{k}{2}\right)_n}{n!^3(1+k)_n^2}(120n^2+84kn+34n+10k+3)$$

The number of exact digits when we sum N terms behaves like

$$\sum_{n=1}^{N} \log_{10}\left[\frac{a(k,n)}{a(k,n+1)}\right] \sim \frac{\ln 16}{\ln 10}N + \left(\frac{1}{2}+3k\right)\frac{\ln N}{\ln 10}$$

Finally I show another formula I have obtained experimentally

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{3n}}{2^{9n}} \frac{\left(\frac{1}{2}-k\right)_n \left(\frac{1}{6}+\frac{k}{3}\right)_n \left(\frac{5}{6}+\frac{k}{3}\right)_n}{n!^2 \left(1+\frac{k}{2}\right)_n} \frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{2}+\frac{k}{3}\right)_n}{\left(\frac{1}{2}+\frac{k}{2}\right)_n \left(\frac{1}{2}\right)_n} R(n,k) = \frac{32\sqrt{2}}{\pi} \frac{2^{3k}}{\binom{2k}{k}}$$

where

$$R(n,k) = \frac{616n^3 + 676n^2 + 214n + 15 + 440n^2k + 312nk + 46k + 88nk^2 + 36k^2 + 8k^3}{(2n+1)(2n+k+1)}$$

Table 1

j	$G_1(n,k)$	CERTIFICATE
1	$\frac{(-1)^n(-1)^k}{2^{6n}2^{2k}}\frac{\binom{2n}{n}^3\binom{2k}{k}^2}{2^{2k}\binom{n-1/2}{k}\binom{n+k}{n}}(4n+1)$	$\frac{4n^2}{(4n+1)(2n-2k-1)}$
2	$\frac{(-1)^k}{2^{8n}2^{2k}}\frac{\binom{2n}{n}^2\binom{2k}{k}\binom{2n+2k}{n+k}}{2^{2k}\binom{n-1/2}{k}}(6n+2k+1)$	$\frac{16n^2}{(6n+2k+1)(2n-2k-1)}$
3	$\frac{(-1)^n(-1)^k}{2^{9n}2^{3k}}\frac{\binom{2n}{n}\binom{n+k}{n}\binom{2n+2k}{n+k}^2}{2^{2k}\binom{n-1/2}{k}}(6n+2k+1)$	$\frac{16n^2}{(6n+2k+1)(2n-2k-1)}$
4	$\frac{(-1)^n(-1)^k}{2^{10n}2^{2k}}\frac{\binom{2n}{n}^2\binom{2k}{k}^2\binom{4n}{2n}}{2^{2k}\binom{2n-1/2}{k}\binom{n+k}{n}}(20n+2k+3)$	$\frac{64n^2}{(20n+2k+3)(2n-2k-1)}$
5	$\frac{(-1)^k3^k}{2^{8n}2^{4k}3^{2n}}\frac{\binom{2n}{n}\binom{4n}{2n}\binom{2k}{k}\binom{2n+2k}{n+k}}{2^{2k}\binom{2n-1/2}{k}}(8n+2k+1)$	$\frac{36n^2}{(8n+2k+1)(2n-2k-1)}$

Table 2

j	$G_1(n,0)$	$G_2(n,0)$
1	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}^3}{2^{6n}}(4n+1) = \frac{2}{\pi}$	$\displaystyle\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^3}{2^{8n}}(6n+1) = \frac{4}{\pi}$
2	$\displaystyle\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^3}{2^{8n}}(6n+1) = \frac{4}{\pi}$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \binom{4n}{2n}\binom{2n}{n}^2}{2^{10n}}(20n+3) = \frac{8}{\pi}$
3	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}^3}{2^{9n}}(6n+1) = \frac{2\sqrt{2}}{\pi}$	$\displaystyle\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}^2\binom{2n}{n}}{2^{12n}} \frac{48n^2+32n+3}{2n+1} = \frac{8\sqrt{2}}{\pi}$
4	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \binom{4n}{2n}\binom{2n}{n}^2}{2^{10n}}(20n+3) = \frac{8}{\pi}$	$\displaystyle\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^3}{2^{12n}}(42n+5) = \frac{16}{\pi}$
5	$\displaystyle\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}\binom{2n}{n}^2}{2^{8n}3^{2n}}(8n+1) = \frac{2\sqrt{3}}{\pi}$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \binom{4n}{2n}\binom{2n}{n}^2}{2^{12n}3^n}(28n+3) = \frac{16\sqrt{3}}{3\pi}$

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Zaragoza (Spain)
E-mail address: jguillera@able.es