CHAINS OF SERIES FOR $1/\pi$ ASSOCIATED TO WZ-PAIRS

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Introduction

If $F(n,k)$ is the first component of a WZ-pair and $F(0,k)=0$ then $F(sn,k+tn)$ with $s \in \mathbb{N}$ and $t \in \mathbb{Z}$ is also the first component of a WZ-pair verifying $F(s\cdot 0, k+t\cdot 0)=0$. We use this remark to obtain chains of series for $1/\pi$.

Chain-1

We consider the function

$$F(n,k)=\frac{-3^k}{2^{16n}2^{4k}}\,\frac{(4n+2k)!(6n+2k)!k!}{(3n+k)!^2(2n+k)!n!^2(2k)!}\,\frac{n^2}{2k+1}$$

$F(n,k)$ leads to

$$\sum_{n=0}^{\infty}\frac{1}{2^{4n}}\,\frac{(1/2)_n(1/4)_n(3/4)_n(1/6)_n(5/6)_n}{(1)_n^3(1/3)_n(2/3)_n}\,\frac{720n^3+804n^2+236n+15}{(n+1/3)(n+2/3)}=\frac{128\sqrt{3}}{\pi}.$$

$F(n,k-n)$ leads to

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{4n}3^n}\,\frac{(1/2)_n(1/4)_n(3/4)_n}{(1)_n^3}(28n+3)=\frac{16\sqrt{3}}{3\pi}.$$

$F(n,k-2n)$ leads to

$$\sum_{n=0}^{\infty}\frac{1}{3^{2n}}\,\frac{(1/2)_n(1/4)_n(3/4)_n}{(1)_n^3}(8n+1)=\frac{2\sqrt{3}}{\pi}.$$

$F(2n,k-3n)$ leads to

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{10n}}\,\frac{(1/2)_n(1/6)_n^2(5/6)_n^2}{(1)_n^3(1/3)_n(2/3)_n}\,\frac{44280n^3+46620n^2+12234n+565}{(n+1/3)(n+2/3)}=\frac{4608\sqrt{3}}{\pi}.$$

$F(2n,k-5n)$ leads to

$$\sum_{n=0}^{\infty}\frac{5^{5n}}{2^{6n}3^{5n}}\,\frac{(1/10)_n(3/10)_n(7/10)_n(9/10)_n}{(1)_n^3(1/2)_n}\,\frac{2924n^2+1668n+105}{n+1/2}=\frac{432\sqrt{3}}{\pi}.$$

Chain-2

We consider the function

$$F(n,k)=\frac{-1}{2^{14n}2^{4k}}\,\frac{(4n+2k)!(6n+2k)!k!}{(3n+k)!(2n+k)!^2n!(2n)!(2k)!}\,\frac{n^2}{2k+1}$$

$F(n,k)$ leads to

$$\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^{3n}\frac{(1/4)_n(3/4)_n(1/6)_n(5/6)_n}{(1)_n^3(1/2)_n}\frac{296n^2+182n+15}{n+1/2}=\frac{128}{\pi}.$$

$F(n,k-n)$ leads to

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}}\frac{(1/2)_n(1/4)_n(3/4)_n}{(1)_n^3}(20n+3)=\frac{8}{\pi}.$$

$F(2n,k-3n)$ leads to

$$\sum_{n=0}^{\infty}\frac{(-1)^n3^{6n}}{2^{12n}}\,\frac{(1/2)_n(1/6)_n^2(5/6)_n^2}{(1)_n^3(1/4)_n(3/4)_n}\frac{7720n^3+8372n^2+2190n+135}{(n+1/4)(n+3/4)}=\frac{2048}{\pi}.$$

Chain-3

We consider the function

$$F(n,k)=\frac{(-1)^n}{2^{14n}2^{2k}}\,\frac{(4n+2k)!^2(2n)!k!}{(3n+k)!(2n+k)!^2n!^3(2k)!}\,\frac{n^2}{2k+1}$$

$F(n,k)$ leads to

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{3n}}\,\frac{(1/2)_n(1/4)_n^2(3/4)_n^2}{(1)_n^3(1/3)_n(2/3)_n}\frac{896n^3+992n^2+296n+21}{(n+1/3)(n+2/3)}=\frac{288}{\pi}.$$

$F(n,k-n)$ leads to

$$\sum_{n=0}^{\infty}\frac{1}{2^{6n}}\,\frac{(1/2)_n^3}{(1)_n^3}(42n+5)=\frac{16}{\pi}.$$

$F(n,k-2n)$ leads to

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}}\,\frac{(1/2)_n(1/4)_n(3/4)_n}{(1)_n^3}(20n+3)=\frac{8}{\pi}.$$

$F(2n,k-3n)$ leads to

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{10n}}\,\frac{(1/2)_n(1/4)_n(3/4)_n(1/6)_n(5/6)_n}{(1)_n^3(1/3)_n(2/3)_n}\frac{29520n^3+31572n^2+8588n+435}{(n+1/3)(n+2/3)}=\frac{6144}{\pi}.$$

Chain-4

We consider the function

$$F(n,k)=\frac{1}{2^{12n}2^{3k}}\,\frac{(4n+2k)!^2k!}{(2n+k)!^3n!^2(2k)!}\,\frac{n^2}{2k+1}$$

$F(n,k)$ leads to

$$\sum_{n=0}^{\infty}\frac{1}{2^{2n}}\,\frac{(1/4)_n^2(3/4)_n^2}{(1)_n^3(1/2)_n}\,\frac{48n^2+32n+3}{n+1/2}=\frac{16\sqrt{2}}{\pi}.$$

$F(n,k-n)$ leads to

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{3n}}\,\frac{(1/2)_n^3}{(1)_n^3}(6n+1)=\frac{2\sqrt{2}}{\pi}.$$

$F(2n,k-n)$ leads to

$$\sum_{n=0}^{\infty}\frac{(-1)^n3^{3n}}{2^{11n}}\,\frac{(1/2)_n(1/6)_n^2(5/6)_n^2}{(1)_n^3(1/3)_n(2/3)_n}\,\frac{29880n^3+32076n^2+8842n+465}{(n+1/3)(n+2/3)}=\frac{4608\sqrt{2}}{\pi}.$$

$F(2n,k-3n)$ leads to

$$\sum_{n=0}^{\infty}\frac{(-1)^n3^{3n}}{2^{9n}}\,\frac{(1/2)_n(1/6)_n(5/6)_n}{(1)_n^3}(154n+15)=\frac{32\sqrt{2}}{\pi}.$$

Chain-5

We consider the function

$$F(n,k)=\frac{1}{2^{10n}2^{2k}}\,\frac{(4n+2k)!(2n+2k)!k!(2n)!}{(n+k)!(2n+k)!^2n!^3(2k)!}\,\frac{n^2}{2k+1}$$

$F(n,k)$ leads to

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}}\,\frac{(1/2)_n(1/4)_n(3/4)_n}{(1)_n^3}(20n+3)=\frac{8}{\pi}.$$

$F(n,k-n)$ leads to

$$\sum_{n=0}^{\infty}\frac{1}{2^{2n}}\,\frac{(1/2)_n^3}{(1)_n^3}(6n+1)=\frac{4}{\pi}.$$

$F(2n,k-3n)$ leads to

$$\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^{3n}\frac{(1/4)_n(3/4)_n(1/6)_n(5/6)_n}{(1)_n^3(1/2)_n}\,\frac{296n^2+182n+15}{n+1/2}=\frac{128}{\pi}.$$

$F(2n, k-n)$ leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{6n}} \frac{(1/2)_n(1/4)_n(3/4)_n(1/6)_n(5/6)_n}{(1)_n^3(1/3)_n(2/3)_n} \frac{9360n^3+10068n^2+2792n+165}{(n+1/3)(n+2/3)} = \frac{2304}{\pi}.$$

Chain-6

We consider the function

$$F(n,k) = \frac{1}{2^{18n}2^{3k}} \frac{(8n+2k)!(4n+2k)!n!k!}{(4n+k)!(3n+k)!(2n+k)!(2n)!n!^2(2k)!} \frac{n^2}{2k+1}$$

$F(n, k-n)$ leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{3n}}{2^{9n}} \frac{(1/2)_n(1/6)_n(5/6)_n}{(1)_n^3}(154n+15) = \frac{32\sqrt{2}}{\pi}.$$

$F(n, k-2n)$ leads to

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{(1/4)_n^2(3/4)_n^2}{(1)_n^3(1/2)_n} \frac{48n^2+32n+3}{n+1/2} = \frac{16\sqrt{2}}{\pi}.$$

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