# About the WZ-pairs which prove Ramanujan series


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**Abstract**

Observing those WZ-demostrable generalizations of the Ramanujan-type series that were already known, we get the insight to make some assumptions concerning the rational parts of those WZ-pairs that prove them. Based on those assumptions, we develop a new strategy in order to prove Ramanujan-type series for $1/\pi$. Using it, we find more WZ-demonstrable generalizations, and so new WZ-proofs, for the 8 Ramanujan-type series already proved, by the WZ-method, in some previous papers by the author. In addition, we discover the first WZ-proofs of three more Ramanujan-type series.


## 1 Introduction

The Ramanujan series for $1/\pi$ are of the form [4, pp. 352-354]

$$\sum_{n=0}^{\infty} z^n \frac{\left(\frac{1}{2}\right)_n (s)_n (1-s)_n}{(1)_n^3}(a+bn) = \frac{1}{\pi}, \tag{1}$$

where $s$ determine the family according to the possible values $s = 1/2,\ 1/4,\ 1/3,$ $1/6$ and the parameters $z$, $a$, $b$ are algebraic numbers such that $-1 \le z < 1$, $a > 0$ and $b > 0$. The symbol $(x)_n$ used in the series is the rising factorial or Pochhammer symbol which is defined for all $x \in \mathbf{C}$ by

$$(x)_n = \begin{cases} x(x+1)\cdots(x+n-1), & n \in \mathbf{Z}^+, \\ 1, & n = 0. \end{cases} \tag{2}$$

Later we will need the more general definition

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}. \tag{3}$$

For $k \in \mathbf{Z} - \mathbf{Z}^-$, (3) coincide with (2). But (3) is more general because it is also defined for all complex $x$ and $k$ such that $x + k \in \mathbf{C} - (\mathbf{Z} - \mathbf{Z}^+)$.

In 1910 Ramanujan discovered 17 series of the type (1), that were published in 1914 in his famous paper [14]. Seventy years later the Ramanujan's work related to these kind of series begun to be understood and since then many other series of this type have been found and proved by using the theory of elliptic modular functions. For example, J. and P. Borwein, in his book [5], prove the 17 series found by Ramanujan. In [7] D. and G. Chudnovsky obtain the fastest possible Ramanujan-type series with a rational value of $z$. In [6] the authors derive some series implicit in the Ramanujan's work. Very recently N. Baruah and B. Berndt in [2] and [3] have got, following closely Ramanujan's ideas as presented in Section 13 of the celebrated paper [14], to prove many new series of this kind with algebraic values of its parameters.

There are 36 Ramanujan series corresponding to rational values of the parameter $z$. They can be written in the form

$$\sum_{n=0}^{\infty} z^n \frac{\left(\frac{1}{2}\right)_n (s)_n (1-s)_n}{(1)_n^3} (a+bn) = \frac{c}{\pi}, \tag{4}$$

where $a$ and $b$ are positive integers and $c^2$ is a rational number. D. Zeilberger in [8] proved the series $s = 1/2$, $z = -1$, $a = 1$, $b = 4$, $c = 2$ in a simple way, as an application of the WZ (Wilf and Zeilberger) method (Sect. 2). In [11] and [12] we used WZ-pairs to obtain WZ-demonstrable generalizations of the series in Table I. In this paper we give a new strategy for the WZ-method in order to prove Ramanujan-type series. It is based on some assumptions concerning two rational functions which allow us to recover all the already known WZ-proofs of Ramanujan-type series.

| $s$ | $z$ | $a$ | $b$ | $c$ | $s$ | $z$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|---|---|---|---|---|
| $1/2$ | $-1$ | $1$ | $4$ | $2$ | $1/2$ | $1/4$ | $1$ | $6$ | $4$ |
| $1/2$ | $-1/8$ | $1$ | $6$ | $2\sqrt{2}$ | $1/2$ | $1/64$ | $5$ | $42$ | $16$ |
| $1/4$ | $-1/4$ | $3$ | $20$ | $8$ | $1/4$ | $1/9$ | $1$ | $8$ | $2/\sqrt{3}$ |
| $1/4$ | $-1/48$ | $3$ | $28$ | $16/\sqrt{3}$ | $1/6$ | $-27/512$ | $15$ | $154$ | $32\sqrt{2}$ |

Table I

Using this strategy, we have found other WZ-demonstrable generalizations of the series in Table I and the first WZ-proofs of those in Table II. We recall that the first proofs of the Ramanujan-type series for $1/\pi$ were based on the theory of modular functions. For example, the formulas in Table I corresponding to $z = 1/4$, $z = 1/64$, $z = -1/4$, $z = -1/48$ and $z = 1/9$ were discovered and proved by S. Ramanujan [14, Eq.: 28, 29, 35, 36, 40]. See also [4, pp. 352-354].

| $s$ | $z$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $1/3$ | $1/2$ | $1$ | $6$ | $3\sqrt{3}$ |
| $1/3$ | $-9/16$ | $1$ | $5$ | $4/\sqrt{3}$ |
| $1/3$ | $-1/16$ | $7$ | $51$ | $12\sqrt{3}$ |

Table II

The first proofs of the identities in Table II were also based on the theory of modular functions but discovered and proved much later [6].

# 2 The WZ-method

We recall [17] that a discrete function $A(n,k)$ is hypergeometric or closed form (CF) if the quotients

$$\frac{A(n+1,k)}{A(n,k)} \quad \text{and} \quad \frac{A(n,k+1)}{A(n,k)}$$

are both rational functions. And a pair of functions $F(n,k)$, $G(n,k)$ is said to be of Wilf and Zeilberger (WZ) if $F$ and $G$ are closed forms and besides

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k). \tag{5}$$

In this case H. S. Wilf and D. Zeilberger [15] have proved that there exists a rational function $C(n,k)$ such that

$$G(n,k) = C(n,k)F(n,k). \tag{6}$$

The rational function $C(n,k)$ is the so-called certificate of the pair $(F,G)$. If a function is a component of a WZ-pair then the Zeilberger's computer package EKHAD [13, Appendix A] obtain the certificate and so the other component. If we sum (5) over all $n \geq 0$, we get

$$\sum_{n=0}^{\infty} G(n,k) - \sum_{n=0}^{\infty} G(n,k+1) = -F(0,k) + \lim_{n\to\infty} F(n,k). \tag{7}$$

So, under the hypothesis

$$\lim_{n\to\infty} F(n,k) = 0,$$

the identity

$$\sum_{n=0}^{\infty} G(n,k) = \sum_{n=0}^{\infty} G(n,k+1),$$

is true if and only if $F(0,k) = 0$.

## 3 Our strategy

We develop an strategy of the WZ-method specifically prepared to obtain generalizations of the Ramanujan-type series

$$\sum_{n=0}^{\infty} z^n B(n)(a+bn) = \frac{c}{\pi} \tag{8}$$

that are of the form

$$\sum_{n=0}^{\infty} z^n y^k B(n,k) R(n,k) = \frac{c}{\pi}, \tag{9}$$

where $y$ is a rational number, $R(n,k)$ is a rational function such that $R(n,0) = a+bn$, and $B(n,k)$ is a closed form such that $B(n,0) = B(n)$. First observe that we know that there exists a rational function $S(n,k)$ such that the couple of closed forms:

$$G(n,k) = z^n y^k B(n,k) R(n,k), \qquad F(n,k) = z^n y^k B(n,k) S(n,k), \tag{10}$$

satisfy the property (WZ pairs):

$$G(n,k+1) - G(n,k) = F(n+1,k) - F(n,k).$$

So, substituting (10) in it, we have

$$\frac{B(n,k+1)}{B(n,k)} R(n,k+1) y - R(n,k) = \frac{B(n+1,k)}{B(n,k)} S(n+1,k) z - S(n,k). \tag{11}$$

We also know $F(0,k) = 0$ and so $n$ is a factor of $S(n,k)$. We will say that the hypergeometric function $B(n,k)$ is good, adequate, etc, to prove a particular Ramanujan's series for $1/\pi$ of type (8), if there are rational numbers $z$, $y$ and a rational function $R(n,k)$ such that (9) is a generalization of it. Our strategy, inspired by a careful observation of the series found in [11] and [12], consists of the following steps: First, we simplify the quotients $B(n+1,k)/B(n,k)$ and $B(n,k+1)/B(n,k)$. Then, we denote the resulting denominators by $P(n,k)$ and $Q(n,k)$, and write them in the form

$$P(n,k) = P_r(n,k) P_{r'}(n,k), \qquad Q(n,k) = Q_s(n,k) Q_{s'}(n,k),$$

where $P_r(n,k)$ and $Q_s(n,k)$ are the polynomial of greatest possible degrees $r$ and $s$ satisfying $P_r(-1,0) \neq 0$ and $Q_s(0,-1) \neq 0$ respectively. Finally, we construct the rational functions $R(n,k)$ and $S(n,k)$ (of degree 1) in the following way

$$R(n,k) = \frac{(a+bn)P_r(n,0) + kU_r(n,q)}{P_r(n,k)}, \qquad S(n,k) = \frac{nV_s(n,k)}{Q_s(n,k)}, \tag{12}$$

where $U_r(n,k)$ and $V_s(n,k)$ are polynomials of degrees $r$ and $s$ with coefficients that we will denote as $d_i$ and $e_i$ respectively. We will determine them in case they exist. For it, we substitute the above expressions of $R(n,k)$ and $S(n,k)$ in (11). Taking $k=-1$ we avoid that $y$ multiplies unknown coefficients, and we get a linear system of equations which allow us to determine the value of $y$. Then, substituting this value of $y$ in (11) and giving values to $k$ and $n$, we can determine all the unknown coefficients $d_i$ and $e_i$. Finally we need to check if the function $B(n,k)$ is a good one substituting everything in (11).

# 4 Our searches

Let us define the functions (by symmetry $j_2 = j_3$ when $s = 1/3, 1/4, 1/6$):

$$C(n,k) = \frac{\left(\frac{1}{2}+j_1k\right)_n (s+j_2k)_n((1-s)_n+j_3k)_n}{(1)_n(1+j_4k)_n(1_n+j_5k)}, \qquad D(k) = \frac{(t)_k(1-t)_k}{(1)_k^2}, \quad (13)$$

where $t$ can be any of the numbers $1/2, 1/3, 1/4, 1/6$ and $j_i$ are adequate rational numbers. The simplest functions $B(n,k)$ that we have tried in this paper are of the form

$$B(n,k) = C(n,k)D(k). \quad (14)$$

In [11], we found some series involving the simpler function

$$D(k) = \frac{\left(\frac{1}{2}\right)_k}{(1)_k}, \quad (15)$$

and there are other candidates for $D(k)$, that we have not tried yet, as for example

$$D(k) = \frac{\left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k \left(\frac{1}{3}\right)_k \left(\frac{2}{3}\right)_k}{\left(\frac{1}{2}\right)_k^2 (1)_k^2}. \quad (16)$$

When we could not find solutions of the form (14), then we tried the longer expression

$$B(n,k) = C(n,k) \cdot \frac{\left(\frac{1}{2}+j_6k\right)_n}{\left(\frac{1}{2}+j_7k\right)_n} \cdot D(k), \quad (17)$$

Of course we can take more factors and so there are many other possibilities. For example, we can consider

$$B(n,k) = C(n,k) \cdot \frac{\left(\frac{1}{4}+j_6k\right)_n \left(\frac{3}{4}+j_6k\right)_n}{\left(\frac{1}{4}+j_7k\right)_n \left(\frac{3}{4}+j_7k\right)_n} \cdot D(k). \quad (18)$$

However we have not made the corresponding searches.

Our searches are not completely random because there are functions that we can discard in advance. For example, the function

$$B(n,k)=\frac{\left(\frac{1}{2}-k\right)_n\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{(1)_n(1+k)_n(1+2k)_n}\cdot\frac{\left(\frac{1}{3}\right)_k\left(\frac{2}{3}\right)_k}{(1)_k^2}, \tag{19}$$

cannot be good because it has an esencial singularity at $k=-1/2$, and so (9) cannot be finite at this value. Another example is the function

$$B(n,k)=\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}+k\right)_n\left(\frac{1}{2}-5k\right)_n}{(1)_n(1+k)_n^2}\cdot\frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2}, \tag{20}$$

which cannot be good because at $k=\frac{1}{10}$ all the summands of (9) are zero except the corresponding to $n=0$ and we can easily check that we do not get a sum of the form $c/\pi$, where $c^2$ is a rational.

## 5 An example of application of our strategy

Using our strategy we prove in detail the following Ramanujan series in Table II

$$\sum_{n=0}^{\infty}\left(\frac{-1}{16}\right)^n\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{(1)_n^3}(51n+7)=\frac{12\sqrt{3}}{\pi}. \tag{21}$$

***Solution 1:*** We will prove that the following function $B(n,k)$ is a good one

$$B(n,k)=\frac{\left(\frac{1}{2}-k\right)_n\left(\frac{1}{2}+k\right)_n\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{\left(\frac{1}{2}+\frac{k}{2}\right)_n\left(1+\frac{k}{2}\right)_n(1+k)_n(1)_n}\cdot\frac{\left(\frac{1}{3}\right)_k\left(\frac{2}{3}\right)_k}{(1)_k^2}. \tag{22}$$

*Proof:* We define the functions

$$G(n,k)=\left(\frac{-1}{16}\right)^n y^k B(n,k)R(n,k), \qquad F(n,k)=\left(\frac{-1}{16}\right)^n y^k B(n,k)S(n,k),$$

and get the evaluations

$$\frac{B(n+1,k)}{B(n,k)}=\frac{(2n-2k+1)(2n+2k+1)(3n+1)(3n+2)}{9(2n+k+1)(2n+k+2)(n+k+1)(n+1)}, \tag{23}$$

$$\frac{B(n,k+1)}{B(n,k)}=-\frac{(2n+k+1)(3k+1)(3k+2)}{9(2n-2k-1)(2n+k+1)(n+k+1)}. \tag{24}$$

Applying our strategy, we can write

$$R(n,k)=\frac{(51n+7)(2n+1)+k(d_1n+d_2k+d_3)}{2n+k+1}$$

and

$$S(n,k)=\frac{n(e_1n+e_2k+e_3)}{2n-2k-1}.$$

To avoid unknown coefficients multiplying $y$, we take $k=-1$ in

$$\frac{B(n,k+1)}{B(n,k)}R(n,k+1)y-R(n,k)=\frac{B(n+1,k)}{B(n,k)}S(n+1,k)z-S(n,k). \tag{25}$$

and, giving values to $n$, we obtain a linear system of equations which allows us to determine $y$ and some of the other unknowns. Substituting, in the general identity with $n$ and $k$, the obtained value of $y$ and giving more values to $n$ and $k$ we can determine all the unknowns $e_1$, $e_2$, $e_3$, $d_1$, $d_2$, $d_3$. In our case we get $y=1$ and the the functions

$$R(n,k)=\frac{(51n+7)(2n+1)+k(90n+24k+28)}{2n+k+1} \tag{26}$$

and

$$S(n,k)=\frac{16n(6n-3k-2)}{2n-2k-1}. \tag{27}$$

Now we check that (25) holds. Then, the WZ method leads to

$$\sum_{n=0}^{\infty}G(n,k)=\sum_{n=0}^{\infty}G(n,k+1),$$

and we can apply the Carlson's theorem (see [1], p. 39) to derive

$$\sum_{n=0}^{\infty}G(n,k)=CONST,$$

where we determine the constant taking $k=1/2$. In this way, we have the formula

$$\sum_{n=0}^{\infty}\left(\frac{-1}{16}\right)^n\frac{\left(\frac{1}{2}-k\right)_n\left(\frac{1}{2}+k\right)_n\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{\left(\frac{1}{2}+\frac{k}{2}\right)_n\left(1+\frac{k}{2}\right)_n(1+k)_n(1)_n} \times\frac{(51n+7)(2n+1)+k(90n+24k+28)}{2n+k+1}=\frac{12\sqrt{3}}{\pi}\cdot\frac{(1)_k^2}{\left(\frac{1}{3}\right)_k\left(\frac{2}{3}\right)_k}. \tag{28}$$

Taking $k=0$ we obtain (21).

***Solution 2:*** Another good function $B(n,k)$ is

$$B(n,k)=\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}+2k\right)_n\left(\frac{1}{3}+k\right)_n\left(\frac{2}{3}+k\right)_n}{\left(\frac{1}{2}+\frac{k}{2}\right)_n\left(1+\frac{k}{2}\right)_n(1+k)_n(1)_n}\cdot\frac{\left(\frac{1}{4}\right)_k\left(\frac{3}{4}\right)_k}{(1)_k^2}. \tag{29}$$

*Proof:* We define the functions

$$G(n,k)=\left(\frac{-1}{16}\right)^n y^k B(n,k)R(n,k), \qquad F(n,k)=\left(\frac{-1}{16}\right)^n y^k B(n,k)S(n,k).$$

Applying our strategy, we obtain $y=1$ and

$$R(n,k)=\frac{(51n+7)(2n+1)+k(114n+36k+37)}{2n+k+1}. \tag{30}$$

and

$$S(n,k)=\frac{-9n(6n^2+30nk+13n-7k-3)}{(3k+1)(3k+2)}, \tag{31}$$

The WZ method leads to

$$\sum_{n=0}^{\infty}G(n,k)=\sum_{n=0}^{\infty}G(n,k+1),$$

and we can apply the Carlson's theorem to derive

$$\sum_{n=0}^{\infty}G(n,k)=CONST,$$

where we determine the constant by substituting $k=-1/3$. We can also find the constant by observing that as a consequence of Weierstrass M-test [16, p. 49], the convergence of the series is uniform. So, the following steps hold

$$\lim_{k\to\infty}\sum_{n=0}^{\infty}G(n,k)=\sum_{n=0}^{\infty}\lim_{k\to\infty}G(n,k)=\frac{18\sqrt{2}}{\pi}\sum_{n=0}^{\infty}\left(\frac{-1}{8}\right)^n\binom{2n}{n}=\frac{12\sqrt{3}}{\pi},$$

where we have used the identity [5, p. 386]:

$$\sum_{n=0}^{\infty}z^n\binom{2n}{n}=\frac{1}{\sqrt{1-4z}}.$$

So, we have proved the following formula

$$\sum_{n=0}^{\infty} \left(\frac{-1}{16}\right)^n \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}+2k\right)_n \left(\frac{1}{3}+k\right)_n \left(\frac{2}{3}+k\right)_n}{\left(\frac{1}{2}+\frac{k}{2}\right)_n \left(1+\frac{k}{2}\right)_n (1+k)_n (1)_n} \times \frac{(51n+7)(2n+1)+k(114n+36k+37)}{2n+k+1} = \frac{12\sqrt{3}}{\pi} \cdot \frac{(1)_k^2}{\left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}. \quad (32)$$

Taking $k=0$ we obtain (21).

# 6 More formulas discovered with our strategy

For each of the indicated Ramanujan series (see Tables I and II), we have chosen an example among the WZ-demonstrable generalizations that we have found using our method.

Case: $s=1/2$, $z=-1$, $a=1$, $b=4$, $c=2$.

$$\sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}-k\right)_n^2 \left(\frac{1}{2}\right)_n}{(1+k)_n^2 (1)_n} (4n+1) = \frac{2}{\pi} \cdot \left(\frac{1}{4}\right)^k \cdot \frac{(1)_k^2}{\left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}. \quad (33)$$

Case: $s=1/2$, $z=1/4$, $a=1$, $b=6$, $c=4$.

$$\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \frac{\left(\frac{1}{2}-k\right)_n \left(\frac{1}{2}+k\right)_n \left(\frac{1}{2}+3k\right)_n}{(1+k)_n (1+2k)_n (1)_n} (6n+6k+1) = \frac{4}{\pi} \cdot \left(\frac{16}{27}\right)^k \cdot \frac{(1)_k^2}{\left(\frac{1}{6}\right)_k \left(\frac{5}{6}\right)_k}. \quad (34)$$

Case: $s=1/2$, $z=-1/8$, $a=1$, $b=6$, $c=2\sqrt{2}$.

$$\sum_{n=0}^{\infty} \left(\frac{-1}{8}\right)^n \frac{\left(\frac{1}{2}-k\right)_n \left(\frac{1}{2}+k\right)_n \left(\frac{1}{2}+3k\right)_n}{(1+k)_n (1+2k)_n (1)_n} (6n+6k+1) = \frac{2\sqrt{2}}{\pi} \cdot \left(\frac{32}{27}\right)^k \cdot \frac{(1)_k^2}{\left(\frac{1}{6}\right)_k \left(\frac{5}{6}\right)_k}. \quad (35)$$

Case: $s=1/4$, $z=-1/4$, $a=3$, $b=20$, $c=8$.

$$\sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \frac{\left(\frac{1}{2}+k\right)_n \left(\frac{1}{4}+\frac{3k}{2}\right)_n \left(\frac{3}{4}+\frac{3k}{2}\right)_n}{(1+k)_n^2 (1)_n} (20n+18k+3) = \frac{8}{\pi} \cdot \left(\frac{16}{27}\right)^k \cdot \frac{(1)_k^2}{\left(\frac{1}{6}\right)_k \left(\frac{5}{6}\right)_k}. \quad (36)$$

Case: $s=1/4$, $z=1/9$, $a=1$, $b=8$, $c=2\sqrt{3}$.

$$\sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^n\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{4}+\frac{3k}{2}\right)_n\left(\frac{3}{4}+\frac{3k}{2}\right)_n}{(1+k)_n^2(1)_n}(8n+6k+1)=\frac{2\sqrt{3}}{\pi}\cdot\frac{(1)_k^2}{\left(\frac{1}{6}\right)_k\left(\frac{5}{6}\right)_k}. \tag{37}$$

Case: $s=1/3$, $z=1/2$, $a=1$, $b=6$, $c=3\sqrt{3}$.

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\frac{\left(\frac{1}{2}+k\right)_n\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{(1+k)_n(1+2k)_n(1)_n}(6n+6k+1)=\frac{3\sqrt{3}}{\pi}\cdot\frac{(1)_k^2}{\left(\frac{1}{3}\right)_k\left(\frac{2}{3}\right)_k}. \tag{38}$$

Case: $s=1/2$, $z=1/64$, $a=5$, $b=42$, $c=16$.

$$\sum_{n=0}^{\infty}\left(\frac{1}{64}\right)^n\frac{\left(\frac{1}{2}-k\right)_n\left(\frac{1}{2}\right)_n\left(\frac{1}{2}+k\right)_n\left(\frac{1}{2}+2k\right)_n}{\left(\frac{1}{2}+\frac{k}{2}\right)_n\left(1+\frac{k}{2}\right)_n(1+k)_n(1)_n}\times\frac{(42n+5)(2n+1)+k(84n+24k+26)}{2n+k+1}=\frac{16}{\pi}\cdot\frac{(1)_k^2}{\left(\frac{1}{4}\right)_k\left(\frac{3}{4}\right)_k}. \tag{39}$$

Case: $s=1/3$, $z=-1/16$, $a=7$, $b=51$, $c=12\sqrt{3}$.

$$\sum_{n=0}^{\infty}\left(\frac{-1}{16}\right)^n\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}+2k\right)_n\left(\frac{1}{3}+k\right)_n\left(\frac{2}{3}+k\right)_n}{\left(\frac{1}{2}+\frac{k}{2}\right)_n\left(1+\frac{k}{2}\right)_n(1+k)_n(1)_n}\times\frac{(51n+7)(2n+1)+k(114n+36k+37)}{2n+k+1}=\frac{12\sqrt{3}}{\pi}\cdot\frac{(1)_k^2}{\left(\frac{1}{4}\right)_k\left(\frac{3}{4}\right)_k}. \tag{40}$$

Case: $s=1/3$, $z=-9/16$, $a=1$, $b=5$, $c=4/\sqrt{3}$.

$$\sum_{n=0}^{\infty}\left(\frac{-9}{16}\right)^n\frac{\left(\frac{1}{2}-k\right)_n\left(\frac{1}{2}+3k\right)_n\left(\frac{1}{3}+k\right)_n\left(\frac{2}{3}+k\right)_n}{\left(\frac{1}{2}\right)_n(1)_n(1+k)_n(1+3k)_n}\times\frac{(5n+1)(2n+1)+k(16n+6k+7)}{2n+1}=\frac{4\sqrt{3}}{3\pi}\cdot 4^k\cdot\frac{(1)_k^2}{\left(\frac{1}{6}\right)_k\left(\frac{5}{6}\right)_k}. \tag{41}$$

Case: $s=1/4$, $z=-1/48$, $a=3$, $b=28$, $c=16\sqrt{3}$.

$$\sum_{n=0}^{\infty}\left(\frac{-1}{48}\right)^n\frac{\left(\frac{1}{2}-k\right)_n\left(\frac{1}{2}+3k\right)_n\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{\left(\frac{1}{2}\right)_n(1)_n(1+k)_n^2}\times\frac{(28n+3)(2n+1)+k(40n+18)}{2n+1}=\frac{16\sqrt{3}}{\pi}\cdot\frac{(1)_k^2}{\left(\frac{1}{6}\right)_k\left(\frac{5}{6}\right)_k}. \tag{42}$$

Case: $s = 1/6$, $z = -27/512$, $a = 15$, $b = 154$, $c = 32\sqrt{2}$.

$$\sum_{n=0}^{\infty} \left(\frac{-27}{512}\right)^n \frac{\left(\frac{1}{2}-k\right)_n \left(\frac{1}{2}+k\right)_n \left(\frac{1}{6}+k\right)_n \left(\frac{5}{6}+k\right)_n}{\left(\frac{1}{2}+\frac{k}{2}\right)_n \left(1+\frac{k}{2}\right)_n (1+k)_n (1)_n} \times \frac{(154n+15)(2n+1)+k(352n+108k+108)}{2n+k+1} = \frac{32\sqrt{2}}{\pi} \cdot \left(\frac{32}{27}\right)^k \cdot \frac{(1)_k^2}{\left(\frac{1}{6}\right)_k \left(\frac{5}{6}\right)_k}. \quad (43)$$

We have checked that our strategy also holds for similar series discovered by the author for the constant $1/\pi^2$ (see [9], [10], [11] and [12]), with the only difference that the degree of the functions $R(n,k)$ and $S(n,k)$ must be two.

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