A class of conjectured series representations for $1/\pi$

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Abstract

Using the second conjecture in the paper [10] of the author and inspired by the theory of modular functions we find a method which allows us to obtain explicit formulae, involving η or θ functions, for the parameters of a class of series for $1/\pi$. As in [10], the series considered in this paper include Ramanujan's series as well as those associated with the Domb numbers and Apéry numbers.

1 A special type of recurrence

The sequence of integer numbers

$$B_n = \frac{(2n)!^3}{n!^6}, \tag{1}$$

satisfies the following recurrence

$$n^3 B_n - 8(2n-1)^3 B_{n-1} = 0.$$

Other sequences of integer numbers satisfying a first order recurrence whose coefficients are third degree polynomials:

$$B_n = \frac{(4n)!}{n!^4}, \tag{2}$$

$$B_n = \frac{(2n)!(3n)!}{n!^5}, \tag{3}$$

and

$$B_n = \frac{(6n)!}{(3n)!n!^3}, \tag{4}$$

satisfy the recursions

$$n^3 B_n - 8(2n-1)(4n-3)(4n-1) B_{n-1} = 0,$$

$$n^3 B_n - 6(2n-1)(3n-2)(3n-1)B_{n-1} = 0$$

and

$$n^3 B_n - 24(2n-1)(6n-5)(6n-1)B_{n-1} = 0,$$

respectively. Examples of sequences of integers which satisfy a second order recurrence with third degree polynomials as coefficients are [1]: The sequence of Domb numbers [5]

$$B_n = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2j}{j} \binom{2n-2j}{n-j}, \tag{5}$$

which satisfy

$$n^3 B_n - 2(2n-1)(5n^2-5n+2)B_{n-1} + 64(n-1)^3 B_{n-2} = 0, \tag{6}$$

the sequence of Apéry numbers

$$B_n = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{j}^2 \tag{7}$$

which satisfy

$$n^3 B_n - (2n-3)(17n^2-17n+5)B_{n-1} + (n-1)^3 B_{n-2} = 0,$$

as well as the sequences

$$B_n = \sum_{j=0}^{n} \binom{n}{j}^4 \tag{8}$$

and

$$B_n = \sum_{j=0}^{\lfloor n/3 \rfloor} 3^{n-3j} \binom{n}{3j} \binom{n+j}{j} \frac{(3j)!}{j!^3}, \tag{9}$$

which satisfy similar recurrences. Our interest on sequences of integers B_n, satisfying a recursion with third degree polynomials as coefficients, comes from the fact that for certain of them [1] there exist algebraic numbers z, a and b such that

$$\sum_{n=0}^{\infty} B_n z^n (a+bn) = \frac{1}{\pi}. \tag{10}$$

Series for $1/\pi$ associated to the sequences in (1), (2), (3) and (4) were first discovered by Ramanujan and had been extensively studied later. Proofs can be found in [2] [3], [4], [7] and [11]. In [9] the author gave simpler proofs of some identities of the form (10) using the WZ-method. Series for $1/\pi$ using the Apéry numbers (7) were presented in a talk of T. Sato [12]. Motivated by them similar series for $1/\pi$ associated with the Domb numbers (5) have been studied and proved in [5]. H. H. Chan [6], also gives some examples of series for $1/\pi$ associated to several sequences, one of them to the numbers (9). Y. Yang has proved similar evaluations [15] using the numbers (8) and following a technique explained in [14]. Other sequences satisfying recurrences with third degree polynomials as coefficients [1] will be used in the examples of section 4.

2 A companion sequence

To each B_n, we associate a companion D_n defined by

$$D_n = \frac{dB_n}{dn},$$

where d/dn means that we differentiate as if n were a continuous variable. From the recurrence of B_n we can obtain a recurrence for D_n. For example, the recurrence (6) for the Domb numbers (5) can be written in the form

$$B_n = 2\frac{(2n-1)(5n^2-5n+2)}{n^3}B_{n-1} - 64\frac{(n-1)^3}{n^3}B_{n-2}$$

and differentiating with respect to n, as if n were a continuous variable, we obtain

$$D_n = 2\frac{(2n-1)(5n^2-5n+2)}{n^3}D_{n-1} - 64\frac{(n-1)^3}{n^3}D_{n-2} +$$

$$6\frac{5n^2-6n+2}{n^4}B_{n-1} - 192\frac{(n-1)^2}{n^4}B_{n-2}.$$

From the initial conditions $B_0 = 1$ and $D_0 = 0$, we get

$$B_1 = 4B_0 + 0B_{-1} = 4, \qquad D_1 = 4D_0 + 0D_{-1} + 6B_0 + 0B_{-1} = 6 \tag{11}$$

and with those values and using the recurrences, we can determine $B_2, B_3, \ldots$ and $D_2, D_3, \ldots$.

3 Two conjectures

In this section we give a method and two conjectures which will allows us to obtain explicit formulae, involving η or θ functions, for the parameters of a class of series for $1/\pi$.

Motivated by the theory of modular functions we begin by introducing the variable

$$q = e^{-\pi\sqrt{N}}. \tag{12}$$

Now, inspired by the paper [10] of the author, we define the functions

$$S(z) = \sum_{n=0}^{\infty} B_n z^n, \tag{13}$$

$$W(z) = \sum_{n=0}^{\infty} \frac{dB_n}{dn} z^n = \sum_{n=0}^{\infty} D_n z^n$$

and consider the following equation relating z and q

$$q = z \exp \frac{W(z)}{S(z)}. \tag{14}$$

If we write z as a series of powers of q

$$z = \alpha_1 q + \alpha_2 q^2 + \alpha_3 q^3 + \alpha_4 q^4 + \cdots. \tag{15}$$

then, the coefficients are given by

$$\begin{aligned} \alpha_1 &= \lim_{z \to 0} \frac{z}{q}, \\ \alpha_2 &= \lim_{z \to 0} \frac{z - \alpha_1 q}{q^2}, \\ \alpha_3 &= \lim_{z \to 0} \frac{z - \alpha_1 q - \alpha_2 q^2}{q^3}. \\ &\vdots \end{aligned} \tag{16}$$

In the same way, if we write S as a series of powers of q

$$S = 1 + \beta_1 q + \beta_2 q^2 + \beta_3 q^3 + \beta_4 q^4 + \cdots, \tag{17}$$

the coefficients are given by

$$\begin{aligned} \beta_1 &= \lim_{z \to 0} \frac{S - 1}{q}, \\ \beta_2 &= \lim_{z \to 0} \frac{S - 1 - \beta_1 q}{q^2}, \\ \beta_3 &= \lim_{z \to 0} \frac{S - 1 - \beta_1 q - \beta_2 q^2}{q^3}. \\ &\vdots \end{aligned} \tag{18}$$

Conjecture 3.1 *The coefficients of (15) and (17), given by (16) and (18), are all integer numbers and z and S are the product of a finite number of Dedekind η functions:*

$$\eta(q) = q^{1/24} \prod_{n=0}^{\infty} (1 - q^{n+1}).$$

Besides, for some rational values of N, z is an algebraic number.

We define the function

$$V(z) = \sum_{n=0}^{\infty} \frac{d}{dn} (B_n z^n) = W(z) + \ln(z) S(z).$$

From (12) and (14), we get the equation

$$\frac{V(z)}{S(z)} = -\pi\sqrt{N} \tag{19}$$

Inspired by the paper [10] of the author, we consider the system

$$\begin{cases} aS + bz\dfrac{dS}{dz} & = & \dfrac{1}{\pi} \\ \\ aV + bz\dfrac{dV}{dz} & = & 0. \end{cases} \tag{20}$$

From (19) and the second equation of (20) we get

$$a(\ln q)S + bz\frac{d}{dz}\left[(\ln q)S\right] = 0. \tag{21}$$

and using (21) we find

$$a(\ln q)S + bz\left[\frac{1}{q}\left(\frac{dz}{dq}\right)^{-1}S + (\ln q)\frac{dS}{dz}\right] = 0. \tag{22}$$

From (22) and the first equation of (20) we obtain the following formula which allows us to determine the parameter b

$$\frac{b}{\sqrt{N}} = \frac{q}{zS}\frac{dz}{dq}. \tag{23}$$

Using the first equation in (20), we get the following formula for the parameter a

$$a = \frac{1}{S}\left(\frac{1}{\pi} - bz\frac{dS}{dz}\right) = \frac{1}{S}\left[\frac{1}{\pi} - bz\frac{dS}{dq}\left(\frac{dz}{dq}\right)^{-1}\right],$$

which, with the use of (23), gives

$$a = \frac{1}{S}\left[\frac{1}{\pi} - \frac{q\sqrt{N}}{S}\frac{dS}{dq}\right], \tag{24}$$

which allow us to determine the parameter a.

Conjecture 3.2 *Substituting the values of z and S in (23) and (24) we obtain values for a and b such that the following identity holds*

$$\sum_{n=0}^{\infty} B_n z^n (a + bn) = \frac{1}{\pi}. \tag{25}$$

Besides, for the rational values of N for which z is an algebraic number (see conjecture 3.1), the parameters a and b are also algebraic numbers.

4 Examples

Example 1

We take the sequence of numbers:

$$B_n = \sum_{j=0}^{n} \binom{2j}{j}^2 \binom{2n-2j}{n-j}^2.$$

The numbers B_n are obtained recursively by setting $B_0 = 1$ and

$$B_n = 8\frac{(2n-1)(2n^2-2n+1)}{n^3} B_{n-1} - 256\frac{(n-1)^3}{n^3} B_{n-2}.$$

Although this is a second order recurrence, we can obtain B_1 as in (11). The companions D_n satisfy the recursion $D_0 = 0$ and

$$D_n = 8\frac{(2n-1)(2n^2-2n+1)}{n^3} D_{n-1} - 256\frac{(n-1)^3}{n^3} D_{n-2} +$$

$$8\frac{6n^2-8n+3}{n^4} B_{n-1} - 768\frac{(n-1)^2}{n^4} B_{n-2}$$

and again, we obtain D_1 as in (11). Following the method described in section 3, we get

$$z = q - 8q^2 + 44q^3 - 192q^4 + 718q^5 - 2400q^6 + 7352q^7 - 20992q^8 + \cdots,$$

$$S = 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 + 96q^6 + 64q^7 + 28q^8 + \cdots.$$

Searching the sequences of the coefficients of these series in *The On-Line Encyclopedia of Integer Sequences* [13], we find that

$$z = \frac{\theta_2^4(q)}{16\theta_3^4(q)} = \frac{\lambda^*(q)^2}{16}, \tag{26}$$

$$S = \theta_3^4(q), \tag{27}$$

where $\theta_2(q)$ and $\theta_3(q)$ are Jacobi theta functions and $\lambda^*(q)$ is the elliptic lambda modulus function, defined by

$$\lambda^*(q) = \frac{\theta_2^2(q)}{\theta_3^2(q)}.$$

Substituting in (13) the values given in (26) and (27), we obtain the formula

$$\theta_3^4(q) = \sum_{n=0}^{\infty} B_n \left(\frac{\theta_2^4(q)}{16\theta_3^4(q)}\right)^n.$$

Substituting (26) and (27) in (23) and expanding in a power series of q, we get

$$\frac{b}{\sqrt{N}} = 1 - 16q + 128q^2 - 704q^3 + 3072q^4 - 11488q^5 + 38400q^6 - \cdots .$$

Again, using *The On-Line Encyclopedia of Integer Sequences* [13], we are lucky and find that

$$\frac{b}{\sqrt{N}} = 1 - \frac{\theta_2^4(q)}{\theta_3^4(q)} = 1 - \lambda^*(q)^2 = \frac{\theta_4^4(q)}{\theta_3^4(q)}. \tag{28}$$

Substituting (27) in (24), we obtain

$$a = \frac{\dfrac{1}{\pi} - 4\sqrt{N}q\dfrac{1}{\theta_3(q)}\dfrac{d\theta_3(q)}{dq}}{\theta_3^4(q)} = \alpha(-q)[1-\lambda^*(q)^2], \tag{29}$$

where $\alpha(q)$ is the elliptic alpha function, defined by

$$\alpha(q) = \frac{\dfrac{1}{\pi} - 4\sqrt{N}q\dfrac{1}{\theta_4(q)}\dfrac{d\theta_4(q)}{dq}}{\theta_3^4(q)}.$$

Substituting in (25) the values of the parameters given in (26), (28) and (29), we obtain the following formula:

$$\frac{1}{\pi} = [1-\lambda^*(q)^2]\sum_{n=0}^{\infty} B_n \left(\frac{\lambda^*(q)^2}{16}\right)^n \left[\alpha(-q) + \sqrt{N}n\right],$$

where $q = e^{-\pi\sqrt{N}}$ or $q = -e^{-\pi\sqrt{N}}$.

Example 2

We take the numbers defined recursively by $B_0 = 1$ and

$$B_n = 4\frac{(2n-1)(3n^2-3n+1)}{n^3}B_{n-1} - 16\frac{(n-1)^3}{n^3}B_{n-2}.$$

The companions D_n satisfy the recursion $D_0 = 0$ and

$$D_n = 4\frac{(2n-1)(3n^2-3n+1)}{n^3}D_{n-1} - 16\frac{(n-1)^3}{n^3}D_{n-2} +$$

$$4\frac{9n^2-10n+3}{n^4}B_{n-1} - 48\frac{(n-1)^2}{n^4}B_{n-2}.$$

Following the method described in section 3, we get

$$z = q - 8q^2 + 28q^3 - 64q^4 + 142q^5 - 352q^6 + 792q^7 - 1536q^8 + 2917q^9 - 5744q^{10} + \cdots$$

and

$$S = 1 + 4q + 8q^2 + 16q^3 + 24q^4 + 24q^5 + 32q^6 + 32q^7 + 24q^8 + 52q^9 + 48q^{10} \cdots$$

With *The On-Line Encyclopedia of Integer Sequences*, we find

$$S = \theta_3^2(q)\theta_3^2(q^2). \tag{30}$$

Using the Maple package *q-series* [8], more specificaty the functions *prodmake* and *etamake*, we find that

$$z = q \prod_{n=0}^{\infty} \left(\frac{1 - q^{2n+1}}{1 - q^{8n+4}} \right)^8 = \left[\frac{\eta(q^8)}{\eta(q^2)} \frac{\eta(q)}{\eta(q^4)} \right]^8 . \tag{31}$$

From the identities [8]

$$\begin{aligned} \theta_2(q) &= 2\frac{\eta^2(q^4)}{\eta(q^2)}, \\ \theta_3(q) &= \frac{\eta^5(q^2)}{\eta^2(q^4)\,\eta^2(q)}, \\ \theta_4(q) &= \frac{\eta^2(q)}{\eta(q^2)}, \end{aligned} \tag{32}$$

we can get

$$\eta(q) = \left[\frac{1}{2}\theta_2(q)\theta_3(q)\theta_4^4(q) \right]^{1/6}, \tag{33}$$

which allows to convert formulas using the Dedekind η function into formulas using the Jacobi θ functions θ_2, θ_3 and θ_4. From (31) and using (33), we can express z with θ functions. A more simplified formula is

$$z = \left[\frac{\theta_2(q^2)}{\theta_2(q)} \frac{\theta_4(q)}{\theta_4(q^2)} \right]^4, \tag{34}$$

which can be obtained by using the first and third identities of (32). Substituting (30) and (34) in (13), we obtain the formula

$$\sum_{n=0}^{\infty} B_n \left[\frac{\theta_2(q^2)}{\theta_2(q)} \frac{\theta_4(q)}{\theta_4(q^2)} \right]^{4n} = \theta_3^2(q)\theta_3^2(q^2).$$

Taking the logarithm of (31) and differentiating with respect to q, we get

$$\frac{q}{z}\frac{dz}{dq} = 1 + 8\sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1-q^{2n+1}} - 8\sum_{n=0}^{\infty} \frac{(8n+4)q^{8n+4}}{1-q^{8n+4}}. \tag{35}$$

From the formula (3.2.24) of [3] and the identities $\theta_4(-q) = \theta_3(q)$ and $\theta_2^4(-q) = -\theta_2^4(q)$, we get

$$\theta_2^4(q) + \theta_3^4(q) = 1 + 24\sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1-q^{2n+1}},$$

which allows us to write (35) using θ functions:

$$\frac{q}{z}\frac{dz}{dq} = \frac{4\theta_2^4(q^4) + 4\theta_3^4(q^4) - \theta_2^4(q) - \theta_3^4(q)}{3}. \tag{36}$$

Substituting (30) and (36) in (23), we obtain

$$\frac{b}{\sqrt{N}} = \frac{4\theta_2^4(q^4) + 4\theta_3^4(q^4) - \theta_2^4(q) - \theta_3^4(q)}{3\theta_3^2(q)\theta_3^2(q^2)}. \tag{37}$$

Substituting (30) in (24), we obtain

$$a = \frac{\dfrac{1}{\pi} - 2\sqrt{N}q\left(\dfrac{1}{\theta_3(q)}\dfrac{d\theta_3(q)}{dq} + \dfrac{1}{\theta_3(q^2)}\dfrac{d\theta_3(q^2)}{dq}\right)}{\theta_3^2(q)\theta_3^2(q^2)}. \tag{38}$$

Substituting in (25) the values of the parameters z, b and a given by (34), (37) and (38), we obtain a family of series for $1/\pi$.

Example 3

We take the numbers defined recursively by $B_0 = 1$ and

$$B_n = 3\frac{(2n-1)(3n^2-3n+1)}{n^3}B_{n-1} + 27\frac{(n-1)^3}{n^3}B_{n-2}.$$

The companions D_n satisfy the recursion $D_0 = 0$ and

$$D_n = 3\frac{(2n-1)(3n^2-3n+1)}{n^3}D_{n-1} + 27\frac{(n-1)^3}{n^3}D_{n-2} +$$

$$3\frac{9n^2-10n+3}{n^4}B_{n-1} + 81\frac{(n-1)^2}{n^4}B_{n-2}.$$

Following the method in section 3, we get

$$\begin{aligned} z &= q-6q^2+9q^3+22q^4-102q^5+108q^6+221q^7-858q^8+810q^9+ \\ &\quad 1476q^{10}-5262q^{11}+4572q^{12}+7802q^{13}-26112q^{14}+21519q^{15}+\cdots \end{aligned}$$

and

$$S = 1+3q+9q^2+12q^3+21q^4+18q^5+36q^6+24q^7+45q^8+12q^9+\cdots$$

Using the Maple package *q-series* [8], more explicitly the functions *prodmake* and *etamake*, we find that

$$z = q\prod_{n=0}^{\infty}\frac{(1-q^{n+1})^6(1-q^{9n+9})^6}{(1-q^{3n+3})^{12}} = \left[\frac{\eta(q)\,\eta(q^9)}{\eta^2(q^3)}\right]^6 \tag{39}$$

and

$$S = \prod_{n=0}^{\infty}\frac{(1-q^{3n+3})^{10}}{(1-q^{n+1})^3(1-q^{9n+9})^3} = \frac{\eta^{10}(q^3)}{\eta^3(q)\,\eta^3(q^9)}. \tag{40}$$

The expressions of z and S allows us to write the formula

$$\sum_{n=0}^{\infty} B_n\left[\frac{\eta(q)\,\eta(q^9)}{\eta^2(q^3)}\right]^{6n} = \frac{\eta^{10}(q^3)}{\eta^3(q)\,\eta^3(q^9)}.$$

And substituting in

$$\sum_{n=0}^{\infty} B_n z^n\left[\frac{1}{S}\left(\frac{1}{\pi}-\frac{q\sqrt{N}}{S}\frac{dS}{dq}\right)+\frac{q\sqrt{N}}{zS}\frac{dz}{dq}n\right] = \frac{1}{\pi}$$

the values of z and S given in (39) and (40), we obtain another family of series for $1/\pi$.

Example 4

It seems that conjecture 3.1 (but not conjecture 3.2) remains true when we consider certain sequences of integers satisfying recurrences whose coefficients are second degree polynomials [1]. As an example we take the sequence of integers [1]

$$B_n = \sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}.$$

This sequence satisfies the recurrence $B_0 = 1$ and

$$B_n = \frac{4(3n^2-3n+1)}{n^2}B_{n-1} - \frac{32(n-1)^2}{n^2}B_{n-2}.$$

The companion numbers D_n satisfy the recurrence $D_0 = 0$ and

$$D_n = \frac{4(3n-2)}{n^3} B_{n-1} - \frac{64(n-1)}{n^3} B_{n-2} +$$

$$\frac{4(3n^2-3n+1)}{n^2} D_{n-1} - \frac{32(n-1)^2}{n^2} D_{n-2}.$$

Following the procedure in section 3, we get

$$z = q - 4q^2 + 12q^3 - 32q^4 + 78q^5 - 176q^6 + 376q^7 - 768q^8 + 1509q^9 - 2872q^{10} + \cdots$$

and

$$S = 1 + 4q + 4q^2 + 4q^4 + 8q^5 + 4q^8 + 4q^9 + 8q^{10} + \cdots.$$

Searching the sequences of the coefficients of these series in *The On-Line Encyclopedia of Integer Sequences* [13], we are lucky and find that

$$z = \left[\frac{\eta^2(q^8)}{\eta(q^4)}\right]^2 \left[-\frac{\eta(q^2)}{\eta^2(-q)}\right]^{-2} = \frac{\theta_2^2(q^2)}{4\theta_3^2(q)}$$

and

$$S = \theta_3^2(q),$$

which allows us to write the formula

$$\sum_{n=0}^{\infty} B_n \left[\frac{\theta_2^2(q^2)}{4\theta_3^2(q)}\right]^n = \theta_3^2(q).$$

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